United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 2

Solutions

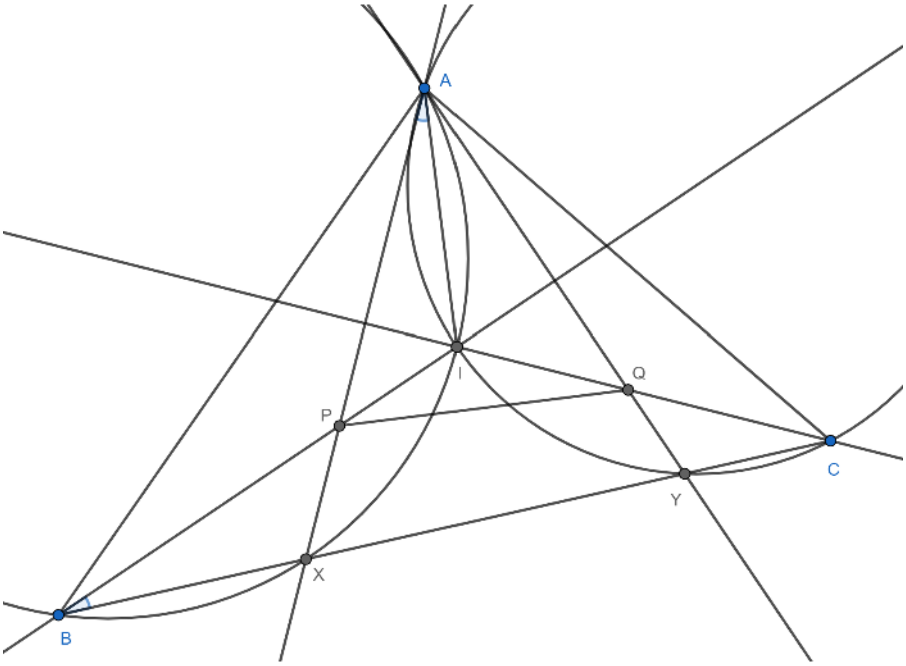
1. Let ABC be a triangle with an obtuse angle A and incentre I. Circles ABI and ACI intersect BC again at X and Y respectively. The lines AX and BI meet at P, and the lines AY and CI meet at Q. Prove that $BCQP$ is cyclic.

Solution

We have $\angle PAI = \angle XBI = \angle IBA$ where the first equality is from circle $ABXI$ and the second from the fact that BI bisects $\angle B$.

This means triangles AIP and BIA are similar, so $IP.IB = IA^2$.

Similarly $IQ.IC = IA^2$ so we are done.



Alternative

We have that:

$$\angle PBA = \angle IBA = \angle XBI = \angle XAI = \angle PAI$$

And so IA is tangent to $\odot ABP$ at I and hence $IA^2 = IP \cdot IB$.

Similarly $IA^2 = IQ \cdot IC$ so $IP \cdot IB = IQ \cdot IC$ which proves $BPQC$ cyclic.

Alternative

There are two natural but hidden circles in this configuration, and discovering them, and then working with one or both of them (in some way), will give rise to various other solutions. The hidden circles are:

(i) The circle with centre I and radius IA. This circle passes through X and Y so $\triangle IXY$ is isosceles with apex A.

(ii) The points $IPXYQ$ lie on a circle.

Here is such an solution:

The figure $BXIA$ is cyclic, so (by angles in the same segment of $\odot BXIA$) $\angle XAI = \angle IXA = B/2$ (and similarly $\angle AYI = C/2$). The exterior angle of the cyclic quadrilateral $XIAB$ at X is $A/2$. By symmetry $\angle IYX$ is also $A/2$. Thus the first hidden circle is manifest, but in this solution we do not use it. Thus $\angle YXA = A/2 + B/2$. The exterior angle of $\triangle YIB$ at I is also $A/2 + B/2$ since that is the sum of its interior opposite angles. Therefore $XYIP$ is cyclic (converse of exterior angle of a cyclic quadrilateral is the interior opposite angle).

For entirely similar reasons $XYQI$ is cyclic, and so $XYQIP$ is a cyclic pentagon.

We want to show that $BCQP$ is cyclic. This is true because $\angle QPI = \angle QYI$ (same segment of $\odot YQIP$) but this is $\angle AYI$ which we proved to be $C/2$. Therefore $\angle QPI = \angle QCB$ and by the converse of exterior angle of a cyclic quadrilateral is the interior opposite angle, it follows that $BCQP$ is cyclic.

Remark

Candidates might attempt to approach the problem by using inversion, and the obvious circle of inversion is Γ_I (the circle with centre I and radius IA). We do not give an inversive proof, but indicate items which may arise in such an attempt. Thanks to the first two solutions, $IP.IB = IA^2 = IQ.IC$, so inversion in Γ_I exchanges P with B and Q with C. The target $\odot BCQP$ is therefore self-inverse and hence orthogonal to Γ_I. Also notice that this inversion leaves $\odot ABP$ and $\odot AQC$ invariant, so they are also orthogonal to Γ_I. This inversion also exchanges $\odot IPXYQ$ with the side line BC.

2. For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible.

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1, 2, 3, 4 \xrightarrow{(i)} 2, 3, 1, 4 \xrightarrow{(ii)} 2, 3, 4, 1$

Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

SOLUTION

For $n \geq 3$ it is straightforward to check by induction that n works if $n \equiv 0$ or 1 modulo 3.

For $n \equiv 2$ modulo 3 we can multiply each number by its position and sum these products. This total, T is invariant modulo 3. We claim that the desired permutation changes T if $n = 3k + 2$. Indeed moving 1 to 2, 2 to 3,... $n - 1$ to n increases T by $\frac{n(n-1)}{2}$ while moving n to 1 decreases T by $n^2 - n$ so overall T decreases by $\frac{n(n-1)}{2} \equiv 1 \quad (3)$.

ALTERNATIVE

$n = 1$ works and for $n = 3$ we take $1, 2, 3 \xrightarrow{(i)} 3, 1, 2$. For $n \geq 4$ we go by induction on n. Suppose. $n \equiv 0, 1 \pmod 3$ then we do:

$$1, 2, \cdots, n-2, n-1, n \xrightarrow{(i)} 1, 2, \cdots, n-3, n, n-2, n-1 \xrightarrow{(ii)} 1, 2, \cdots, n-4, n, n-3, n-2, n-1$$

The rules about moves only care about numbers modulo 3 so as we can send:

$$1, 2, \cdots, n-4, n-3 \to n-3, 1, 2, \cdots, n-4$$

by the inductive hypothesis we can send:

$$1, 2, \cdots, n-2, n-1, n \to 1, 2, \cdots, n-4, n, n-3, n-2, n-1 \to n, 1, 2, \cdots, n-4, n-3, n-2, n-1$$

as $n - 3 \equiv n \equiv 0, 1 \pmod 3$. This completes the induction.

We now show that we can't achieve this for $n \equiv 2 \pmod 3$.

We work modulo 3 denoting numbers on the board by $1, 2, 3$ dependent on their remainder. Let P_1, P_2, P_3 denote the number of pairs $(1, 2)$, $(2, 3)$, $(3, 1)$ respectively that occur on the board in that order. We claim the move leaves $P = P_1 + P_2 + P_3$ invariant.

For *(ii)* this is clear as the two numbers we switch have the same labels.

For *(i)* observe that if two of the numbers have the same label then so must the third in which case P is invariant. If the numbers all have distinct labels we have two cases:

$$(1, 2, 3) \to (3, 1, 2) \to (2, 3, 1) \to (1, 2, 3)$$
$$(1, 3, 2) \to (2, 1, 3) \to (3, 2, 1) \to (1, 3, 2)$$

Note that this move only affects the number of pairs involving two numbers in our cycle. We can check that in the first case there are always 2 such pairs and in the second case there is always 1 such pair which proves the claim.

Notice that the difference in P between the sequences $1, 2, \ldots, n$ and $n, 1, 2, \ldots, n-1$ arises only from pairs involving n. As $n \equiv 2 \pmod 3$, the number of such pairs in the first sequence is:

$$\#\{1 \leq m \leq n-1 : m \equiv 1 \pmod 3\} = \frac{n-2}{3} + 1 = \frac{n+1}{3}$$

For the second sequence the number of such pairs is:

$$\#\{1 \leq m \leq n-1 : m \equiv 3 \pmod 3\} = \frac{n-2}{3}$$

These are not equal which shows we cannot go between the two sequences.

3. For an integer $n \geq 3$, we say that $A = (a_1, a_2, \ldots, a_n)$ is an *n-list* if every a_k is an integer in the range $1 \leq a_k \leq n$. For each $k = 1, \ldots, n-1$, let M_k be the minimal possible non-zero value of $\left|\frac{a_1+\ldots+a_{k+1}}{k+1} - \frac{a_1+\ldots+a_k}{k}\right|$, across all n-lists. We say that an n-list A is *ideal* if

$$\left|\frac{a_1+\ldots+a_{k+1}}{k+1} - \frac{a_1+\ldots+a_k}{k}\right| = M_k$$

for each $k = 1, \ldots, n-1$.

Find the number of ideal n-lists.

Solution

We observe that $M_k = \frac{1}{k(k+1)}$; it is certainly of the form $\frac{n}{k(k+1)}$ for some integer n, and the sequence 2,1,1,1,1,... shows $n = 1$ can be attained.

Next we observe that the first two terms of an ideal sequence must be consecutive integers.

If the first three terms of an ideal sequence are $a, a+1, x$ or $a+1, a, x$ then $\frac{2a+1+x}{3} - \frac{2a+1}{2} = \pm\frac{1}{6}$ so $2x = 2a + 1 \pm 1$ so $x = a$ or $a + 1$.

Changing notation slightly, if we call the third term in an ideal sequence, c then we know that the first two terms sum to $2c + \varepsilon$ for $\varepsilon \in \{-1, +1\}$. We now claim that all terms from the third onward are equal to c. For the base of the induction there is nothing to prove, so we assume that the terms 3 to k are all equal to c and denote term $k+1$ by x.

We have $\dfrac{kc+\varepsilon+x}{k+1} - \dfrac{kc+\varepsilon}{k} = \pm\dfrac{1}{k(k+1)}$

that is $k(kc+\varepsilon+x) - (k+1)(kc+\varepsilon) = \pm 1$ or $kx = kc + \varepsilon \pm 1$.

The only way that $\varepsilon \pm 1$ can be a multiple of k is if it is zero, so $x = c$ as claimed.

To finish the problem we must count the sequences. We choose a consecutive pair of integers ($n-1$ ways), choose whether the first two terms of the sequence will be increasing or decreasing (2 ways), and finally choose whether the third term will equal the smaller or the larger of the first two (2 ways). Thus there are $4(n-1)$ possible sequences.

Alternative

Firstly it's easy to see $M_k = \frac{1}{k(k+1)}$.

Let $s_k = a_1 + \cdots + a_k$ then, considering the two possible signs for the expression inside the absolute value, we see a sequence is ideal iff:

$$a_{k+1} \in \left\{\frac{s_k+1}{k}, \frac{s_k-1}{k}\right\}$$

For this to give integer choices for a_{k+1}, we want $s_k \equiv \pm 1 \pmod{k}$. This is true for $k = 1$ and, if we choose a_{k+1} in the above set then we have:

$$a_{k+1} \equiv \frac{s_k \pm 1}{-1} \equiv -s_k \mp 1 \pmod{k+1} \implies s_{k+1} \equiv \mp 1 \pmod{k+1}$$

So this condition will be automatically satisfied.

Observe that for $k \geq 3$, exactly one of the choices will be an integer and this integer will be in the required ranges because

$$0 < \frac{2-1}{k} \leq \frac{s_k - 1}{k} < \frac{s_k + 1}{k} \leq \frac{nk+1}{k} < n+1$$

So after choosing (a_1, a_2, a_3), there is precisely one way to continue the sequence.

We have n choices for a_1 and then two choices for both a_2 and a_3. The only way these can fail to be valid is if they fall outside the range $1 \leq a_k \leq n$. With a bit of thought, we see this only fails for:

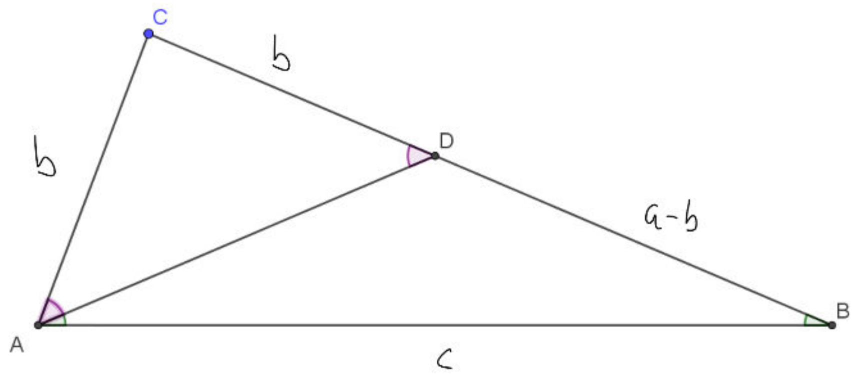
$$(1,0,0)\,,\ (1,0,1)\,,\ (n, n+1, n+1)\,,\ (n, n+1, n)$$

So there are $(n \times 2 \times 2) - 4 = \boxed{4(n-1)}$ sequences in total.

4. The side lengths a, b, c of a triangle ABC are positive integers such that the highest common factor of a, b and c is 1. Given that $\angle A = 3\angle B$ prove that at least one of a, b and c is a cube.

SOLUTION

Let $A = 3B$ and let D be on BC such that BAD is isosceles with apex D.



Angles $\angle DAC$ and $\angle CDA$ are both $2\angle B$ (using the condition in the question and the external angles in a triangle respectively), so $\triangle DAC$ is isosceles with apex C.

This means $DB = a - b$ so $2(a - b)\cos(B) = c$.

The cosine rule gives $(a - b)(a^2 + c^2 - b^2) = ac^2$.

Now $a^3 - a^2b - ab^2 + b^3 = bc^2$, so $a^3 = b(a^2 + ab - b^2 + c^2)$.

If some prime p divides b then it divides a^3 and thus a. This means it cannot divide c since a, b and c are coprime. It therefore does not divide $(a^2 + ab - b^2 + c^2)$.

This means $v_p(b) = v_p(a^3)$ for all primes dividing b, so b is a cube.

ALTERNATIVE

Obtain the relation $bc^2 = (a + b)(a - b)^2$ and use the substitution $y = \frac{c}{a}$, $x = \frac{b}{a}$ to write it as $y^2 = (x + 1)(x - 1)^2$.

This is a singular elliptic curve – see the double point at $(1, 0)$ – that can be rationally parametrised. If p, q are coprime numbers then consider the line $y = \frac{p}{q}(x - 1)$ – through $\left(0, -\frac{p}{q}\right)$ and $(1, 0)$ – and its third intersection point with the curve:

$$\frac{a}{b} = x = \frac{p^2 - q^2}{q^2}, \qquad \frac{c}{b} = y = \frac{(p^2 - 2q^2)p}{q^3},$$

where the condition $a > b$ is equivalent to $p > \sqrt{2}q$. Here $p^2 - q^2$ and q^2 are coprime and also $(p^2 - 2q^2)p$ and q^3 therefore

$$a = (p^2 - q^2)q, \qquad b = q^3, \qquad c = (p^2 - 2q^2)p.$$

ALTERNATIVE

Let the angles be $\theta, 3\theta, \pi - 4\theta$. Since the lengths are all integers, the sine rule tells us that $\frac{\sin(3\theta)}{\sin\theta}$ and $\frac{\sin \pi - 4\theta}{\sin\theta}$ are both rational. Some angle sum invocations on the 3θ term reveal that $\cos^2\theta$ must be rational, and similar on the 4θ term shows that $\cos\theta$ is also rational, which implies the earlier condition. So the sides have rational ratios iff $\cos\theta$ is rational. Let $\cos\theta = \frac{m}{n}$, in lowest terms, we then get the ratios

$$1 : 4\frac{m^2}{n^2} - 1 : 8\frac{m^3}{n^3} - 4\frac{m}{n}$$

Clearing the denominators, we get ratios

$$n^3 : 4m^2n - n^3 : 8m^3 - 4mn^2$$

This is clearly in lowest terms, therefore the first length is a cube.